Das Wohltemperirte Clavier

Pitch, Tuning and Temperament Design

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Introduction

The suggestion that the sinuous circles of J. S. Bach's 1722 cover sheet for *Das Wohltemperirte Clavier* may depict a tuning scheme was made some years ago by Andreas Sparschuh, a mathematician from the Technical University, Darmstadt[1]. In an award winning presentation to the German Tuners Association, Sparschuh outlined his numerical reading of Bach's design:

Start-1-1-1-0-0-0-2-2-2-2-2-End

The clavichord specialist Michael Zapf and the tuning theorist Herbert Anton Kellner, both independently met with Sparschuh. While Zapf was convinced that Bach's diagram was indeed a tuning system, he was sceptical regarding Sparschuh's initial interpretation, considering it at odds with historical tuning practice. Zapf subsequently made a proposal of his own, treating Sparschuh's numbers as an equal-beating

[1] Andreas Sparschuh, *Deutsche Mathematiker Vereinigung Jahrestagung*, 1999.

specification in terms of seconds-per-beat[2]. Keith Biggs from BT Labs investigated the mathematics of Zapf's proposal and noted that, by closing the circle-of-fifths, the pitch of Bach's keyboard might be determined[3]. A subsequent proposal by the author[4], interpreted Sparschuh's numbers as beats-per-second, and included both ends of Bach's diagram in the analysis to close the circle-of-fifths[5]:

{1}-1-1-1-0-0-0-2-2-2-2-2-{2}

Shortly after that, a proposal by Lehman[6] interpreted Sparschuh's numbers as 1/12 and 1/6 of a Pythagorean comma, but without a consistent reading for the end points[7].

Tuning Circumstances at Bach's Time

Lorentz Christoph Mizler's main claim to fame today is as the founder of the society whose 14th member was Johann Sebastian Bach. On the subject of tuning, he once noted that while Werckmeister's temperament was the best in his day, it had nevertheless been improved upon since Neidhardt's time[8]. However, we know that the best tuning practices of Mizler's day did not satisfy Bach; for as his son Carl Philipp Emanuel noted, no one else could tune the harpsichord of his father to his satisfaction[9]. In this regard, Lindley proposes[10] that the elaborate theoretical tuning models expounded by the likes of Neidhardt and Sorge, were unable to capture the subtle nuances that Bach customarily achieved in tuning[11]. He, suggests, moreover,

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[2] Zapf's temperament derivation is available to members of his Clavichord Discussion Group at: http://launch.groups.yahoo.com/group/clavichord/files

[3] Keith Biggs, 'Letter to the Editor', *Early Music Review*, May 2003.

[4] John Charles Francis, 'The Esoteric Keyboard Temperaments of J. S. Bach', *Eunomious*, Feb. 2005.

[5] Strictly speaking, two circles, as will be seen.

[6] Bradley Lehman, 'Bach's extraordinary temperament: our Rosetta Stone', Vol. xxxiii, No. 1 & No. 2, *Early Music*, 2005.

[7] For his reading, Lehman turns Bach's diagram upside-down, explaining this curiosity as a facet of Bach's pedagogic method.

[8] Lorenz Christoph Mizler, *Neu eröffnete mus. Bibliothek oder gründliche Nachricht nebst unparteiischen Urteil v. mus. Schriften u. Büchern*, vol. 1, Part 3, Leipzig 1737, p. 55.

[9] Carl Philipp Emanuel Bach, *Biographische Mitteilungen über Johann Sebastian Bach.*

[10] Mark Lindley, 'A Quest for Bach's Ideal Style of Organ Temperament', *Stimmungen im 17. und 18. Jahrhundert : Vielfalt oder Konfusion?* Stiftung Kloster Michaelstein, *1997, p. 45-67* .

[11] Lindley notes in this context, C. P. E Bach's remark that his father was not much given to theory.

that the cause of the theoretical deficiency was that Sorge and Neidhardt would never split their basic unit of measurement for tempering, namely 1/12 of a Pythagorean comma. Lindley notes that this fraction is unable to satisfy the following three conditions simultaneously:

1. A ***gradual***[12] variation of the major and minor thirds (or sixths)
2. The smallest major 3rd C-E beats larger than pure[13]
3. The most heavily tempered thirds are impure by less than a syntonic comma (i.e. Pythagorean Thirds are excluded)

We will see that the tuning for *Das Wohltemperirte Clavier* achieves all of these goals and that, as Lindley intimates, in so doing a finer division of the Pythagorean comma is needed.

Marpurg, while arguing the case for Equal Temperament, noted that there is only one ideal version of this temperament, whereas there are many unequal schemes[14]. He therefore suggested there would be chaos until Equal Temperament was universally achieved, as otherwise each would be inclined to use whatever they considered best. We will see that in *Das Wohltemperirte Clavier*, J. S. Bach offers a tuning system that is also an ideal of its kind.

J. S. Bach's use of the generic term 'Clavier' (keyboard) leaves unspecified the precise instrument to be used. As noted by Richard Jones[15], this, in conjunction with the deliberately circumscribed keyboard compass, suggests that the work was intended to be universally accessible to keyboard players regardless of the particular type of instrument (harpsichord, clavichord, or organ[16]) that they might have at their

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[12] The emphasis is that of Lindley.

[13] In Bach's case, we have testimony from Kirnberger as reported by Wilhelm Friederich Marpurg in his *Versuch über die musikalische Temperatur*, Breslau 1766, p. 213.

[14] Marpurg, op. cit., p. 194.

[15] Richard Jones, *Oxford Composer Companions : J. S. Bach*, Ed. Malcolm Boyd, 1999, p. 516.

[16] Bach took issue with the temperament of at least one organ builder. For Andreas Sorge, commenting in 1748 on the tuning system of the organ builder Gottfried Silbermann, mentions Bach as having described four specific triads resulting from Silbermann's method as having a barbaric nature intolerable to a good ear (*Bach-Dokumente II*, no. 575). Edward John Hopkins, also relates an anecdote whereby J. S. Bach as auditor of Silbermann's instruments said "You tune the organ in the manner you please, and I play the organ in the key I please"; following his remark with a Fantasy in A-flat major causing Silbermann to retire to avoid his own "wolf" (*The Organ*, London, 1855, p.143.).

disposal. An important issue arises here, however, in relation to the adoption of an unequal temperament for 'well-tempered' instruments; namely coexistence at cammerton and cornet-ton pitch[17]. In the case of equal tempering, this is no concern as each key is identical. However, if unequally tempered instruments at cammerton and cornet-ton pitch are tuned identically to an identical temperament, then there will be inevitable intonation problems as the tuning of one instrument is two places removed from the other on the circle-of-fifths. It follows that there cannot be just one temperament for *Das Wohltemperirte Clavier*, given the generic term Bach uses, but rather there must be two variants: one for cammerton and another for cornet-ton. We will see that the temperament for *Das Wohltemperirte Clavier* consists of two transposed cammerton, cornet-ton variants, indicated by respective left-to-right and right-to-left readings of Bach's diagram.

By virtue of certain telling features in the cammerton and cornet-ton tuning variants, Bach's scheme can be shown to be an equal-beating one. Its transposition by two places on the circle-of-fifths requires adaptation to one of the beat rates, which Bach explicitly provides. While the equal-beating methods presented by Jorgensen have been viewed cautiously in some circles, they are nevertheless pertinent to Bach's method. Jorgensen, in his impressive tome, *Tuning*, traces the historical understanding of beats, noting that in the Seventeenth Century there was no information published that they should increase in frequency when intervals are played higher up the scale[18]. To the musicians at that time, he notes, the quality of the fifths in a scale when played harmonically, seemed identical when their beat frequencies were the same; deceiving them into believing that tempered fifths were of the same size when their beat frequencies were identical[19]. Nor, it seems, had the situation changed by the end of the Eighteenth Century, for in the context of Thomas Young's rules for well temperament of 1799, Jorgensen notes that it would be authentic practice to apply the much easier equal-beating methods[20].

C. P. E. Bach writing in 1753 concerning the tuning of the clavichord and harpsichord[21] refers to tempering *most*[22] of the fifths. He observed that the beats of

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[17] Bruce Haynes, *A History of Performing Pitch*, Scarecrow Press, 2002.

[18] Owen Jorgensen, *Tuning*, Michigan State University Press, 1991.

[19] Jorgensen refers here to intervals within an octave.

[20] Jorgensen also cites Roger North's tuning instructions of 1726, where he revealed some professional tuners were listening to the beats of fifths, which should beat at the same speed as slow quavers.

[21] Carl Philipp Emmanuel Bach, *Versuch über die wahre Art das Clavier zu spielen*, 1753, p. 10.

fifths may be more easily heard by probing fourths[23]. In minimalist terms, he characterised unequal well-tempered tuning:

> Both types of instrument must be tempered as follows: In tuning the fifths and fourths, testing minor and major and chords, take away from most of the fifths a barely noticeable amount of their absolute purity. All twenty-four tonalities will thus become usable. The beats of fifths can be more easily heard by probing fourths, an advantage that stems from the fact that the tones of the latter lie closer together than fifths. [24]

Clearly, this generic description does not indicate which fifths should be tempered, nor by how much. Moreover, although one is admonished to listen for the beats of fifths and fourths, there is no indication as to how fast those beats must be in order to take away from the fifth a barely noticeable amount of its absolute purity.

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[22] Most, but not all, so excluding Equal Temperament.

[23] This remark reflects an understanding that a tempered fifth and its inversion downwards as a fourth must beat identically.

[24] «Beyde Arten von Instrumenten müssen gut temperirt seyn, indem man durch die Stimmung der Qvinten, Qvarten, Probirung der kleinen und grossen Tertien und gantzer Accorde, den meisten Qvinten besonders so viel von ihrer größten Reinigkeit abnimmt, daß es das Gehör kaum mercket und man alle vier und zwantzig Ton=Arten gut brauchen kan. Durch Probirung der Qvarten hat man den Vortheil, daß man die nöthige Schwebung der Qvinten deutlicher hören kan, weil die Qvarten ihrem Grund=Ton näher liegen als die Qvinten. Sind die Claviere so gestimmt, so kan man sie wegen der Ausübung mit Recht für die reinste Instrumente unter allen ausgeben, indem zwar einige reiner gestimmt aber nicht gespielet werden. Auf dem Claviere spielet man aus allen vier und zwantzig Ton=Arten gleich rein und welches wohl zu mercken vollstimmig, ohngeachtet die Harmonie wegen der Verhältnisse die geringste Unreinigkeit sogleich entdecket. Durch diese neue Art zu temperiren sind wir weiter gekommen als vor dem, obschon die alte Temperatur so beschaffen war, daß einige Ton=Arten reiner waren als man noch jetzo bey vielen Instrumenten antrift. Bey manchem andern Musico würde man vielleicht die Unreinigkeit eher vermercken, ohne einen Klang-Messer dabey nöthig zu haben, wenn man die hervorgebrachten melodischen Töne harmonisch hören solte. Diese Melodie betrügt uns oft und läßt uns nicht eher ihre unreinen Töne verspüren, bis diese Unreinigkeit so groß ist, als kaum bey manchem schlecht gestimmten Claviere.»

J. S. Bach's Tuning Procedures

The diagram on the cover sheet of *Das Wohltemperirte Clavier,* read left-to-right, indicates a tuning scheme for cammerton-pitched instruments. Bearings can be set by proceeding around the circle-of-fifths towards the flats using only fourths and fifths, so as to remain within the octave starting on *Middle C*[25]. Bach's diagram indicates three types of interval: i) intervals beating once per second, ii) intervals beating twice per second and iii) pure intervals without beats. In the Eighteenth Century, such tempi could be readily determined from a pocket watch or pendulum clock; today, a metronome at 60 beats per minute will achieve the same result. The protrusion at the left of Bach's diagram depicts the beat-rate of the interval closing the circle-of-fifths, representing a check that the temperament has been set correctly. That interval arises as a side effect of the tuning scheme, and it beats exactly once per second. A realisation of the tuning procedure constrained to one octave, is illustrated in Figure 1, yielding a temperament denoted as *R2-1*[26]. An alternative formulation, yielding the same temperament, is presented in Figure 2[27].

Instruments at cornet-ton pitch sound one tone higher than their cammerton counterparts and, to coexist harmoniously in an ensemble, their temperament must be transposed downwards by a whole tone. Read right-to-left, Bach's diagram yields a transposed tuning scheme for cornet-ton-pitched instruments. Bearings are set by proceeding around the circle-of-fifths towards the sharps, the protrusion at the right of Bach's diagram giving the expected beat rate for the implicitly tuned interval closing the circle-of-fifths as two beats per second. A realisation of the tuning procedure within one octave is detailed in Figure 3, yielding a temperament denoted as *R12-2*[28].

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[25] This is the octave referenced by the opening notes of the *Praeludium in C* BWV 846.

[26] The author's previous article introduced this nomenclature.

[27] This procedure uses only one beat rate rather than two; that rate being exactly one beat per second (i.e., 60 beats per minute). Either the fourth or fifth may be used to hear the beats, as suggested by C. P. E. Bach. At each tuning step, the octave is set from a previously tuned note, and then the intervening note is tuned such that the fourth and/or fifth beat once per second. It is vital that such beating is achieved by narrowing the fifth, or equivalently by widening the fourth.

[28] David Griffel first suggested some years ago that the 'C' in 'Clavier' might be a pitch reference for a right-to-left reading of Bach's diagram. This idea has been followed-up in a previous paper by the author and most recently by Lehman. While the hook on the 'C' is a commonly found artefact in Bach-related manuscripts, it is indeed possible that Bach intentionally placed the 'C' to show the right-to-left cornet-ton solution commences on F rather than the more usual C.

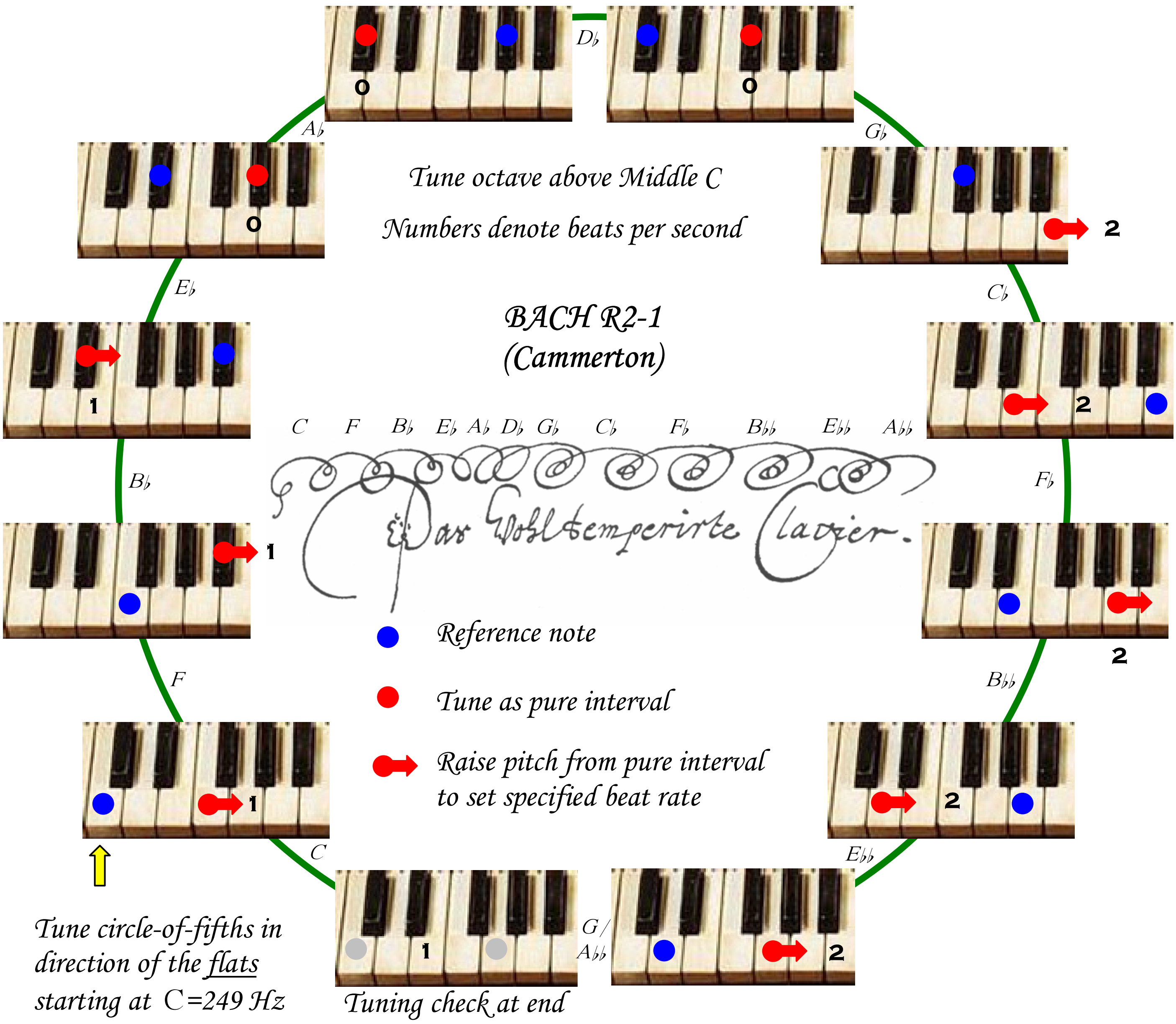

Figure 1: Canonical form of the cammerton tuning procedure

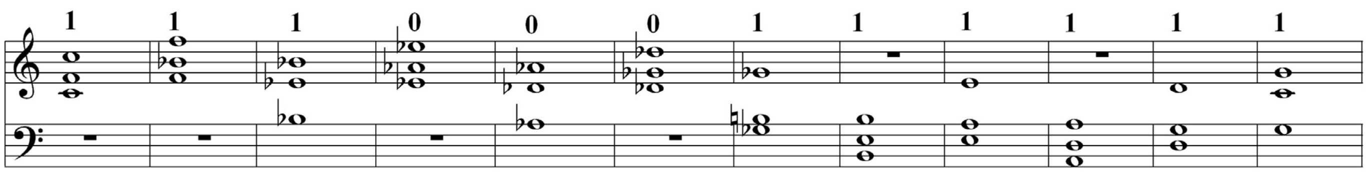

Figure 2: Optimised cammerton tuning procedure using single beat rate of 1 beat per second[29]

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[29] All octaves, as well as fifths marked ‘0’, should be tuned pure. The fifths marked ‘1’ are narrowed by raising the pitch of the lower note until they beat once per second. The fourth can also be used to set or detect these beat rates, as the indicated fourths and fifths beat identically when the octave is pure.

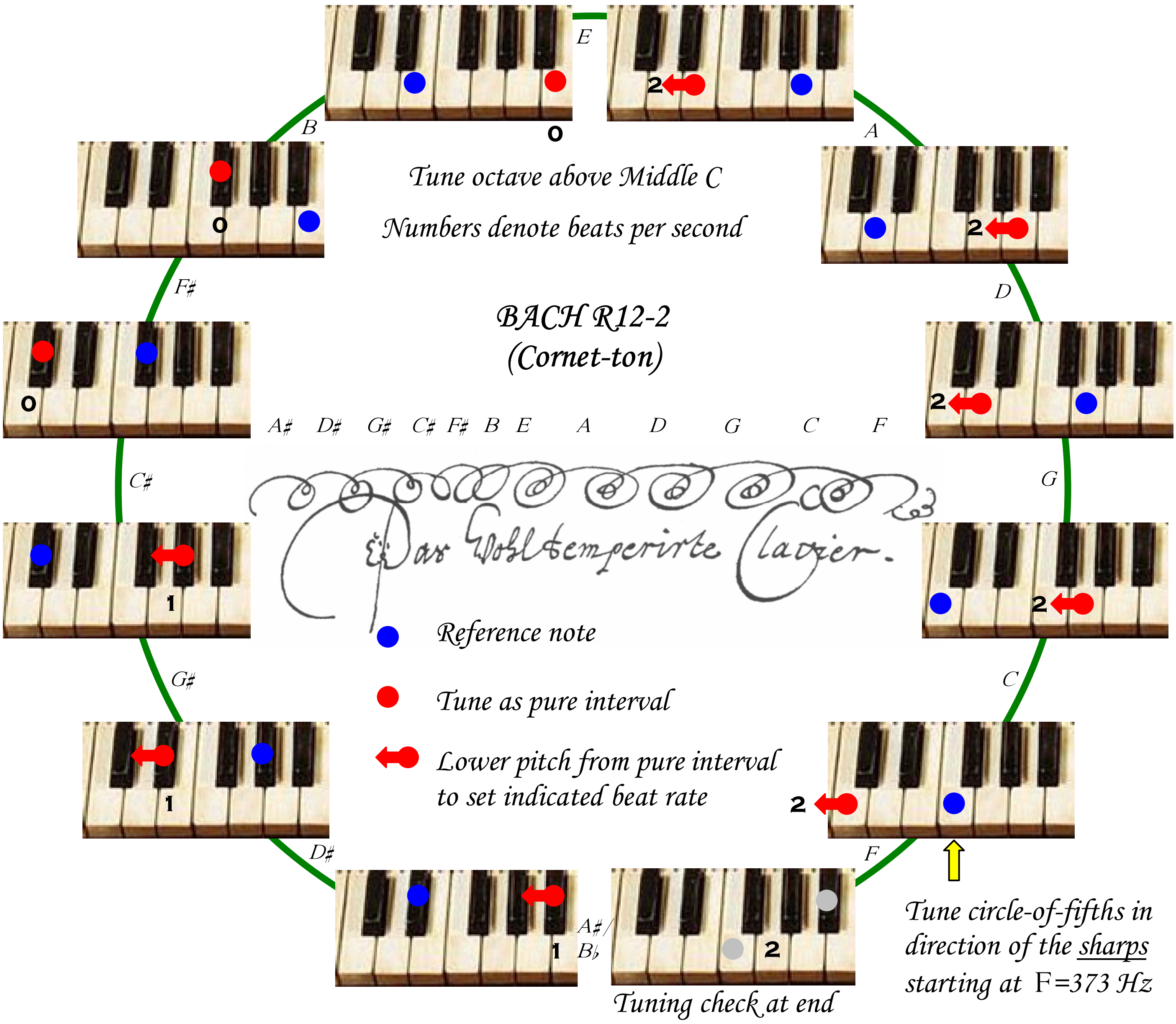

Figure 3: ***Canonical form of the cornet-ton tuning procedure***[30]

Pitch and Temperament

The pitch at which Bach's tuning procedures were designed can be obtained by reverse-engineering his diagram using the indicated beat rates (see appendices for details). The corresponding temperaments can then be derived from this pitch information. The pitch and temperament are summarised in Table 1 and Table 2, respectively, proving that the circles in Figure 1 and Figure 3 are exact transpositions.

[30] Optimisations can be made in like manner to Figure 2.

R2-1 (Cammerton)	**Frequency (Hz)**	**R12-2 (Cornet-ton)**
C	249.072	
D♭	263.154	
D	279.331	C
E♭	296.049	D♭
E	312.998	D
F	332.43	E♭
G♭	350.873	E
G	373.109	F
A♭	394.732	G♭
A	417.997	G
B♭	443.573	A♭
B	468.497	A
	498.145	B♭
	526.309	B

Table 1: Pitch of *Das Wohltemperirte Clavier*[31]

[31] Simple enharmonic spellings are used from now on.

R2-1 (Cammerton)			R12-2 (Cornet-ton)		
Note	**Temperament (Cents)**	**Deviation From Equal Temperament (Cents)**	**Note**	**Temperament (Cents)**	**Deviation From Equal Temperament (Cents)**
C	0	0			
D♭	95.2136	- 4.8			
D	198.495	- 1.5	C	0	0
E♭	299.124	- 0.9	D♭	100.628	0.6
E	395.505	- 4.5	D	197.010	- 3.0
F	499.782	- 0.2	E♭	301.286	1.3
G♭	593.259	- 6.7	E	394.763	- 5.2
G	699.637	- 0.4	F	501.141	1.1
A♭	797.169	- 2.8	G♭	598.673	- 1.3
A	896.314	- 3.7	G	697.818	- 2.2
B♭	999.128	- 0.9	A♭	800.633	0.6
B	1093.770	- 6.2	A	895.273	- 4.7
			B♭	1001.500	1.5
			B	1096.720	- 3.3

Table 2: Temperament of *Das Wohltemperirte Clavier*[32]

[32] The cammerton and cornet-ton temperaments are exact transpositions.

Temperament Design

The temperament for *Das Wohltemperirte Clavier* arises naturally as an ideal providing an optimally smooth progression from worst to best thirds across the circle-of-fifths, so satisfying Lindley's first condition. The thinking that led to its creation can be readily reconstructed from the well known consideration that four consecutive intervals on the circle-of-fifths determine the width of the corresponding major third. The Pythagorean third (~408 cents) is excluded, so that a sequence of four pure fifths does not occur, so satisfying Lindley's second condition. The widest third must then consist of three consecutive pure intervals on the circle-of-fifths, followed by a tempered interval to narrow the third. This observation yields the sequence *0-0-0-1* on the circle-of-fifths, corresponding to the worst-case third. Optimising for a gradual progression towards better thirds as Lindley suggests, leads to a progressive narrowing of the major thirds by one unit at a time. The results in Table 3 are now predicated.

Tempering of intervals on circle-of-fifths	Narrowing from Pythagorean Third
(0, 0, 0, 1)	1 unit
0, (0, 0, 1, 1)	2 units
0, 0, (0, 1, 1, 1)	3 units
0, 0, 0, (1, 1, 1, 1)	4 units
0, 0, 0, 1, (1, 1, 1, 2)	5 units
0, 0, 0, 1, 1, (1, 1, 2, 2)	6 units
0, 0, 0, 1, 1, 1, (1, 2, 2, 2)	7 units
0, 0, 0, 1, 1, 1, 1, (2, 2, 2, 2)	8 units
0, 0, 0, 1, 1, 1, 1, 2, (2, 2, 2, 2)[33]	8 units

Table 3: Derivation of *Das Wohltemperirte Clavier* tuning by progressive narrowing of thirds

Considering the resulting sequence *0, 0, 0, 1, 1, 1, 1, 2, 2, 2, 2, 2,* it can be seen that the sum of the digits is 14[34]. Accordingly, as the Pythagorean comma must be distributed over the circle-of-fifths, each tempering unit theoretically corresponds to

[33] Including a *0* rather than the final *2* yields a Pythagorean third, while including a *1* rather than the final *2* breaks the pattern of grouping similar fifths, and requires a less convenient beat rate for tuning.

[34] Based on principles of gematria, "BACH" equates to 14.

1/14th part of a Pythagorean comma, the tempering fractions being 1/7[35] and 1/14. Adopting the historical precedent of placing the best thirds in the key of C, yields the theoretic temperament R12-14P in Figure 4. The equal-beating method for tuning this temperament (R12-2) is depicted in Bach's diagram, and realised in Figure 3[36].

F C G D A E B F# C# G# D# A# (E#/F)

-1/7 -1/7 -1/7 -1/7 -1/7 0 0 0 -1/14 -1/14 -1/14 -1/14

Figure 4: R12-14P - theoretic representation of R12-2 in terms of Pythagorean comma fractions

The cornet-ton temperament in Figure 4 must be transposed by a whole tone for cammerton-pitched instruments to give R2-14P as shown in Figure 5.

G D A E B F# C# G# D# A# E# B# (F##/G)

-1/7 -1/7 -1/7 -1/7 -1/7 0 0 0 -1/14 -1/14 -1/14 -1/14

Figure 5: R2-14P - theoretic representation of R2-1 in terms of Pythagorean comma fractions

The equal-beating method for tuning this temperament (R2-1) is depicted in Bach's diagram, and has been realised in Figure 1 and Figure 2.

The difference in beat rates between cammerton (R2-1) and cornet-ton (R12-2) tuning procedures that occurs at the interval closing the circle-of-fifths, is explicitly represented in Bach's diagram, and can easily be derived as indicated in Figure 6 from the following well-understood tuning principles:

1. An interval without beats in a given octave is without beats in the octave above.
2. Inverting a fourth (e.g. C1-F1) such that its lower note sounds an octave higher (i.e. F1-C2 for the given example), yields an interval which beats at the same rate.
3. Inverting a fifth (e.g. C1-G1) such that its lower note sounds an octave higher (i.e. G1-C2 for the given example), yields an interval that beats twice as fast.

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[35] Historical precedents for 1/7-comma usage include the Werckmeister *Septenarium* temperament.

[36] The equal-beating tuning method uses a rate of one beat per second to represent 1/14-comma tempering and two beats per second to represent 1/7-comma adjustments, where the requisite intervals are in the octave starting on Middle C. When transposed an octave lower, intervals beating twice per second will beat once per second, so 1/7-comma adjustments can also be interpreted as an octave shift downwards, where the interval beats once per second.

Cammerton	C	D♭	D	E♭	E	F	G♭	G	A♭	A	B♭	B	C	D♭	*Beat Rate*
Loop 1	C					F									*1*
Loop 2						F					B♭				*1*
Loop 3				E♭							B♭				*1*
Loop 4				E♭					A♭						*0*
Loop 5		D♭							A♭						*0*
Loop 6		D♭					G♭								*0*
Loop 7							G♭					B			*2*
Loop 8					E							B			*2*
Loop 9					E					A					*2*
Loop 10			D							A					*2*
Loop 11			D					G							*2*
Left End	C							G							*1*
Cammerton	C	D♭	D	E♭	E	F	G♭	G	A♭	A	B♭	B	C	D♭	
Cornet-ton			C	D♭	D	E♭	E	F	G♭	G	A♭	A	B♭	B	
Loop 1						E♭							B♭		*1*
Loop 2						E♭					A♭				*1*
Loop 3				D♭							A♭				*1*
Loop 4				D♭					G♭						*0*
Loop 5									G♭					B	*0*
Loop 6							E							B	*0*
Loop 7							E					A			*2*
Loop 8					D							A			*2*
Loop 9					D					G					*2*
Loop 10			C							G					*2*
Loop 11			C					F							*2*
Right End								F					B♭		*2*

Figure 6: Comparison of the cammerton and cornet-ton tuning procedures. Intervals in green cross the octave without changing beat rate, as in one case a fourth is inverted upwards and in two other instances the interval is pure. However, when the interval in red crosses the octave it doubles beat rate since a fifth is inverted upwards. The respective beat rates of 1 and 2 are captured by the protrusions at the left and right of Bach's diagram.

Properties of the Derived Tuning Schemes

The properties of the cornet-ton tuning scheme based on 1/14–comma fractions is now compared with the equal beating implementation derived from Bach's diagram. Figure 9 gives the width of the major thirds, showing a smooth progression from best thirds in C-E and F-A to the widest third at E-G#. The equal beating scheme tracks the theoretic one closely generally matching within one cent, an exception being E♭-G, where the equal beating tuning is almost two cents narrower than theory.

Figure 8 shows the corresponding situation for minor thirds; the best occuring in A-C and D-A and progressively narrowing until D♭-E. As before, the tracking of the equal beating tuning is generally within one cent, a minor exception being the third at G-B♭.

Figure 9 shows the width of the fifths; the smallest occurring at C and F, while the largest are at B, G♭ and D♭.

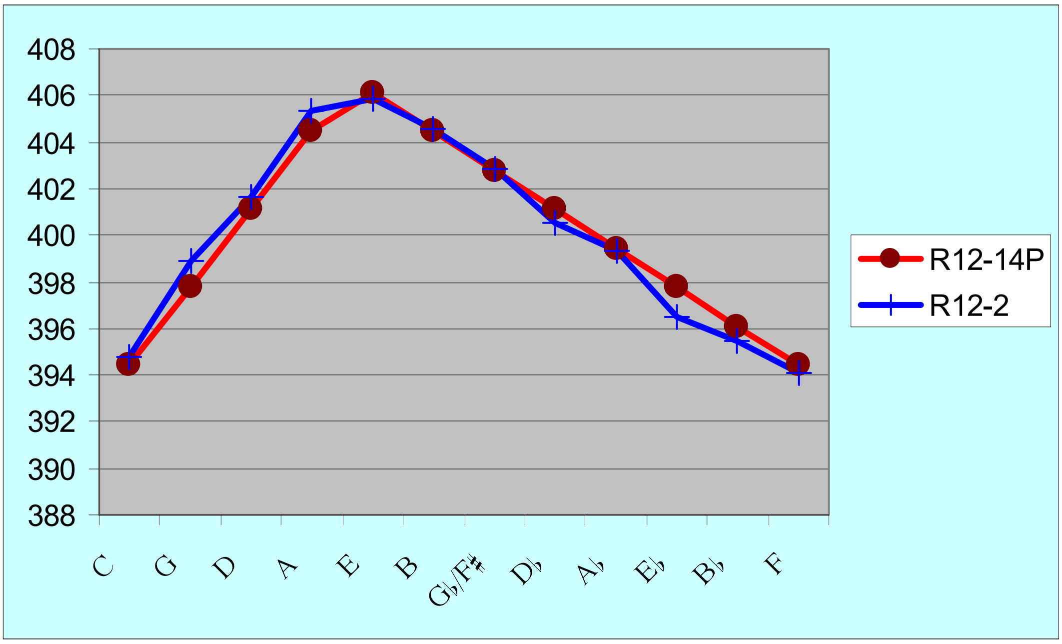


Figure 7: Width of major thirds in cents for R12-14P (theoretic) and R12-2 (equal beating)

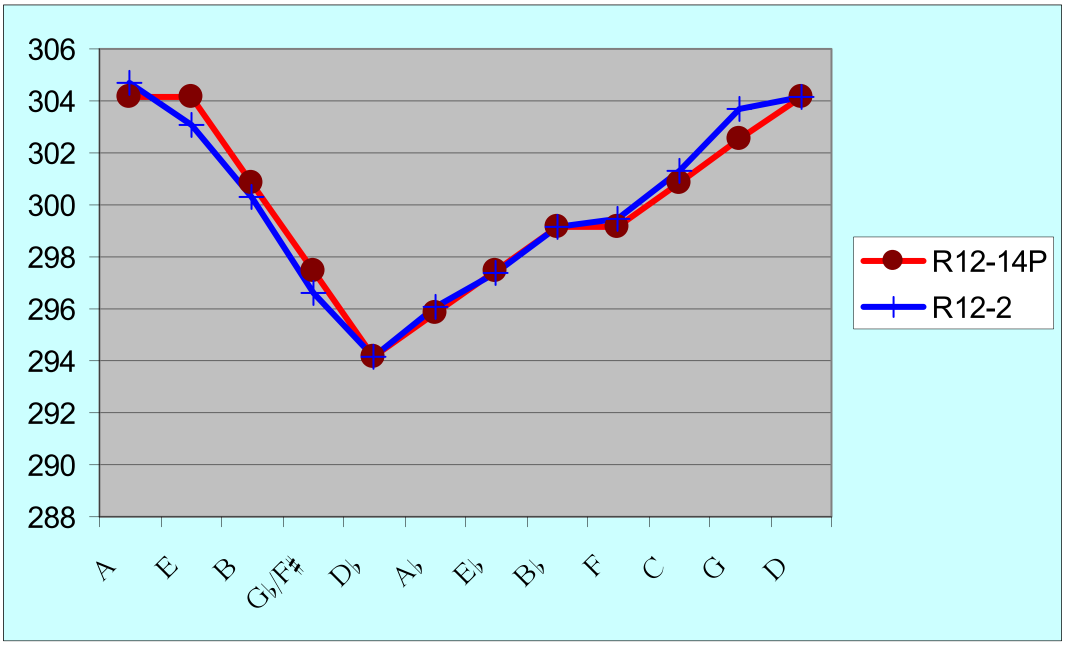


Figure 8: Width of minor thirds in cents for R12-14P (theoretic) and R12-2 (equal beating)

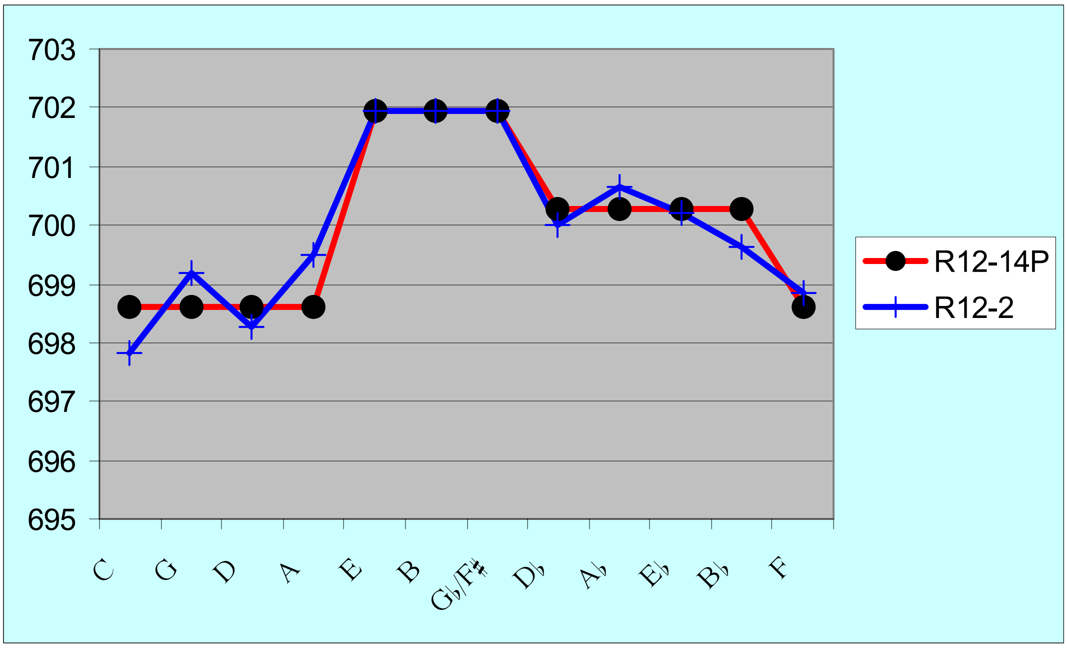


Figure 9: Width of fifths in cents for R12-14P (theoretic) and R12-2 (equal beating)

Comparison with other Temperaments

A comparison of R12-2 (cornet-ton) with other temperaments is given in terms of the Euclidian distance in cents in Figure 10 and in terms of the correlation distance in Figure 11. For the cammerton temperament R2-1, the comparison in terms of Euclidian distance in cents is given in Figure 12, while Figure 13 shows the correlation distance. In terms of correlation distance, the closest match to R12-2 is

Zapf, followed by Lehman and Sorge[37]. For R2-1, the closest matches in terms of correlation distance are Neidhardt Circulating No. 1, Sparschuh, Young No. 1, Mercadier and Barnes.

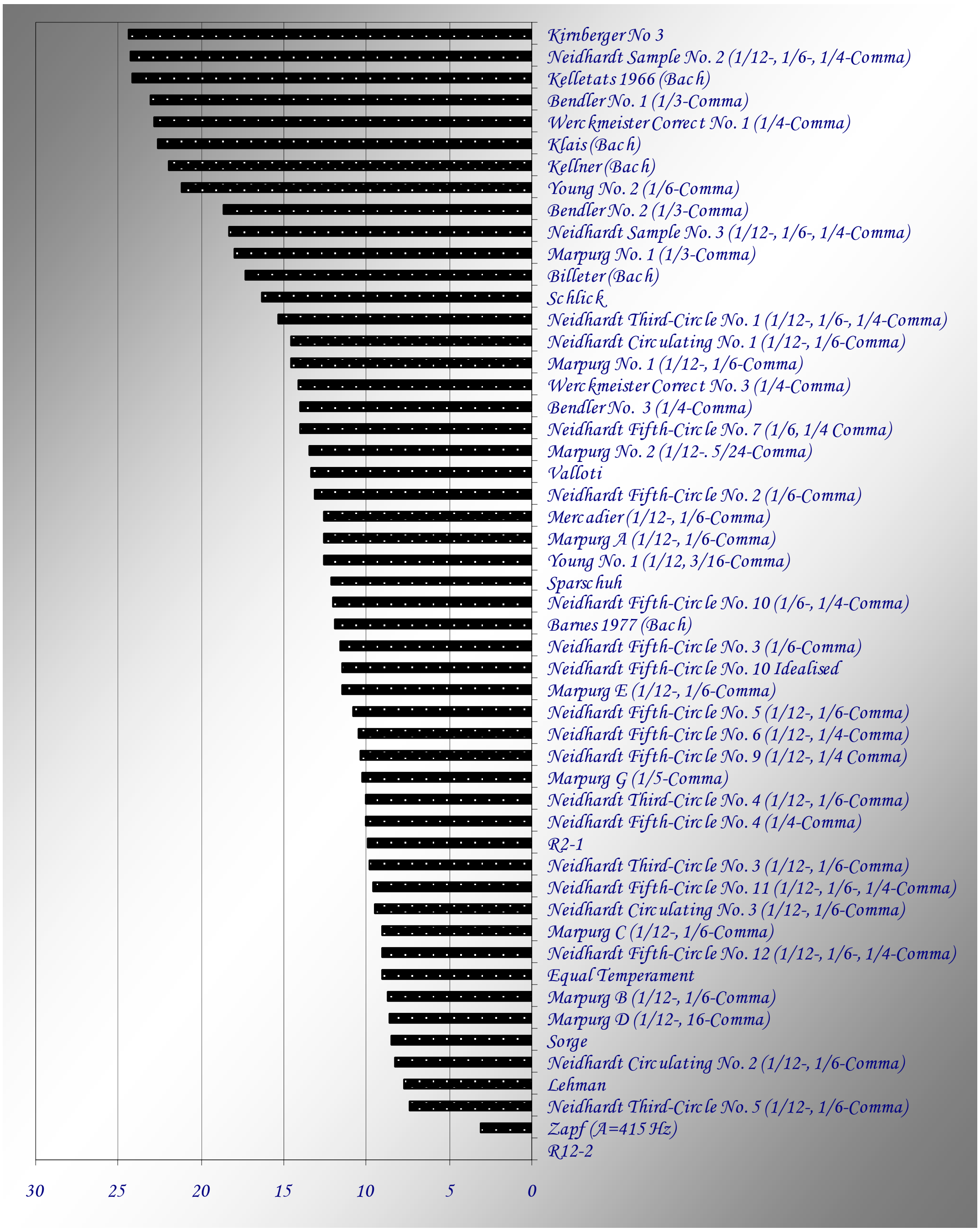


Figure 10: Comparison of R12-2 with other temperaments (Euclidian distance in cents)[38]

[37] Georg Andreas Sorge (1703-1778) was the fifteenth member of the Mizler Society, joining immediately after Bach. The cornet-ton temperament may have been preserved on the Leipzig organs as mentioned by Lehman, and a speculative scenario is that Sorge copied it.

[38] The historical temperaments above were largely derived from J. Murray Barbour's, *Tuning and*

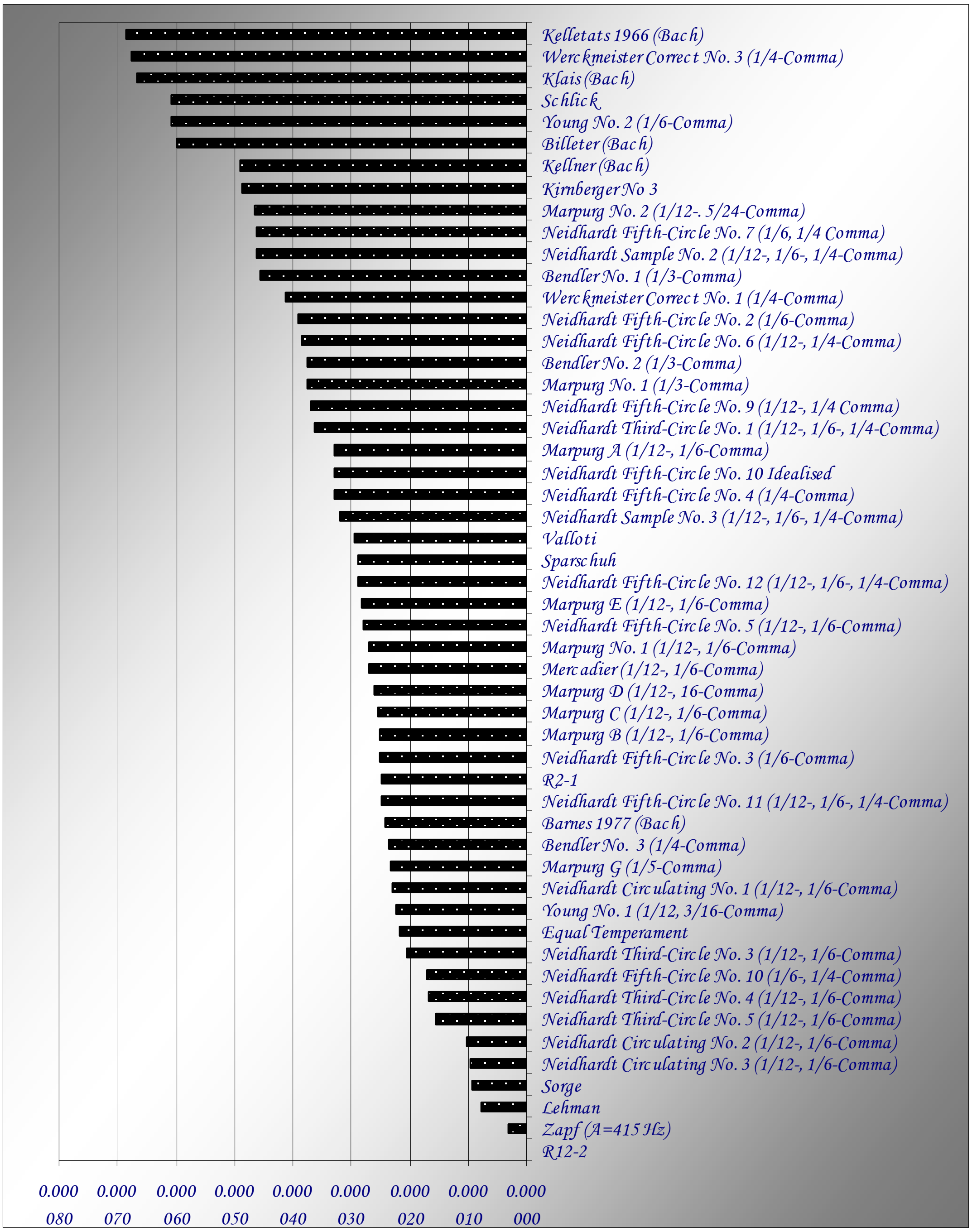


Figure 11: Comparison of R12-2 with other temperaments (correlation distance)

Temperament, Dover, 2004.

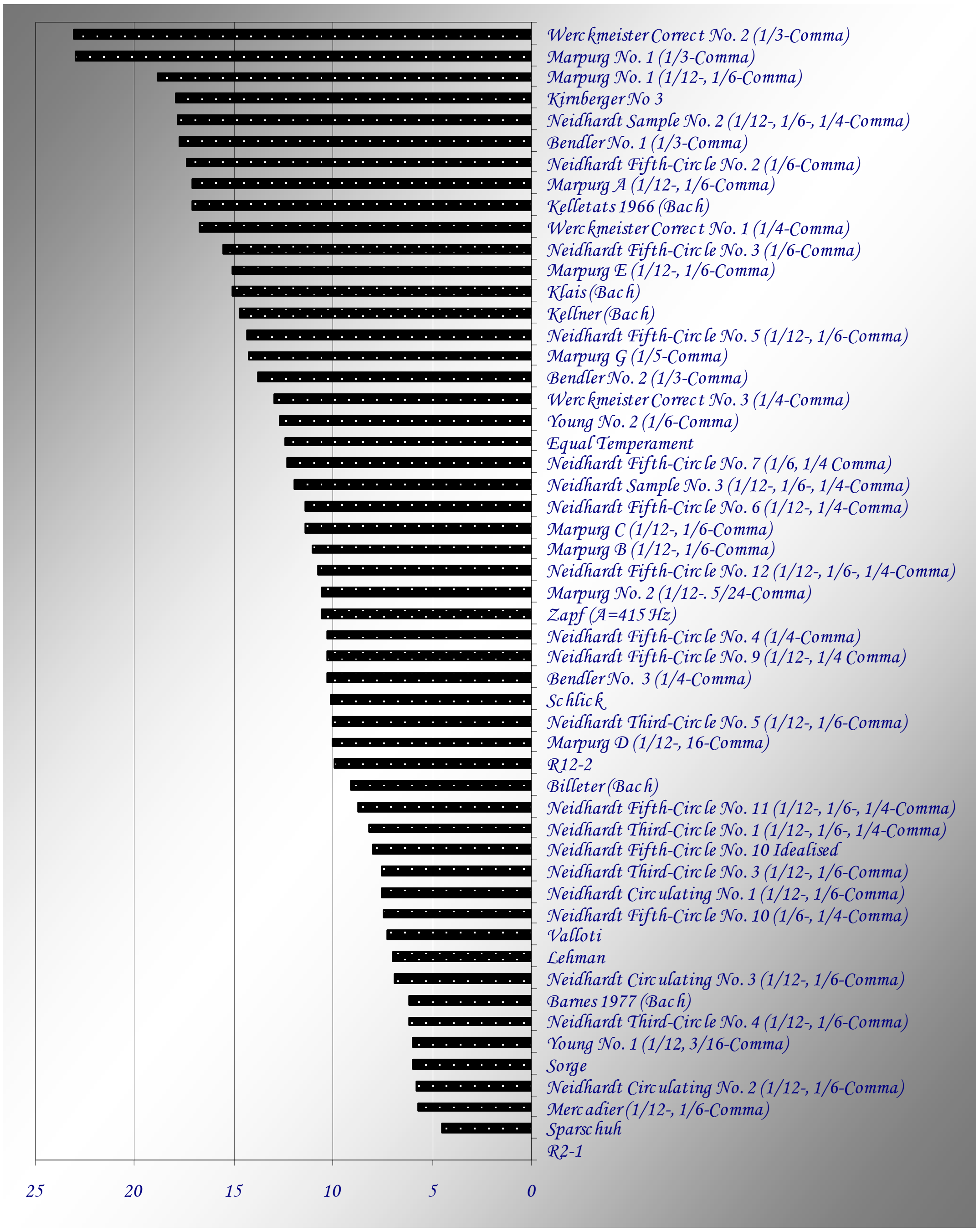


Figure 12: Comparison of R2-1 with other temperaments (Euclidian distance in cents)

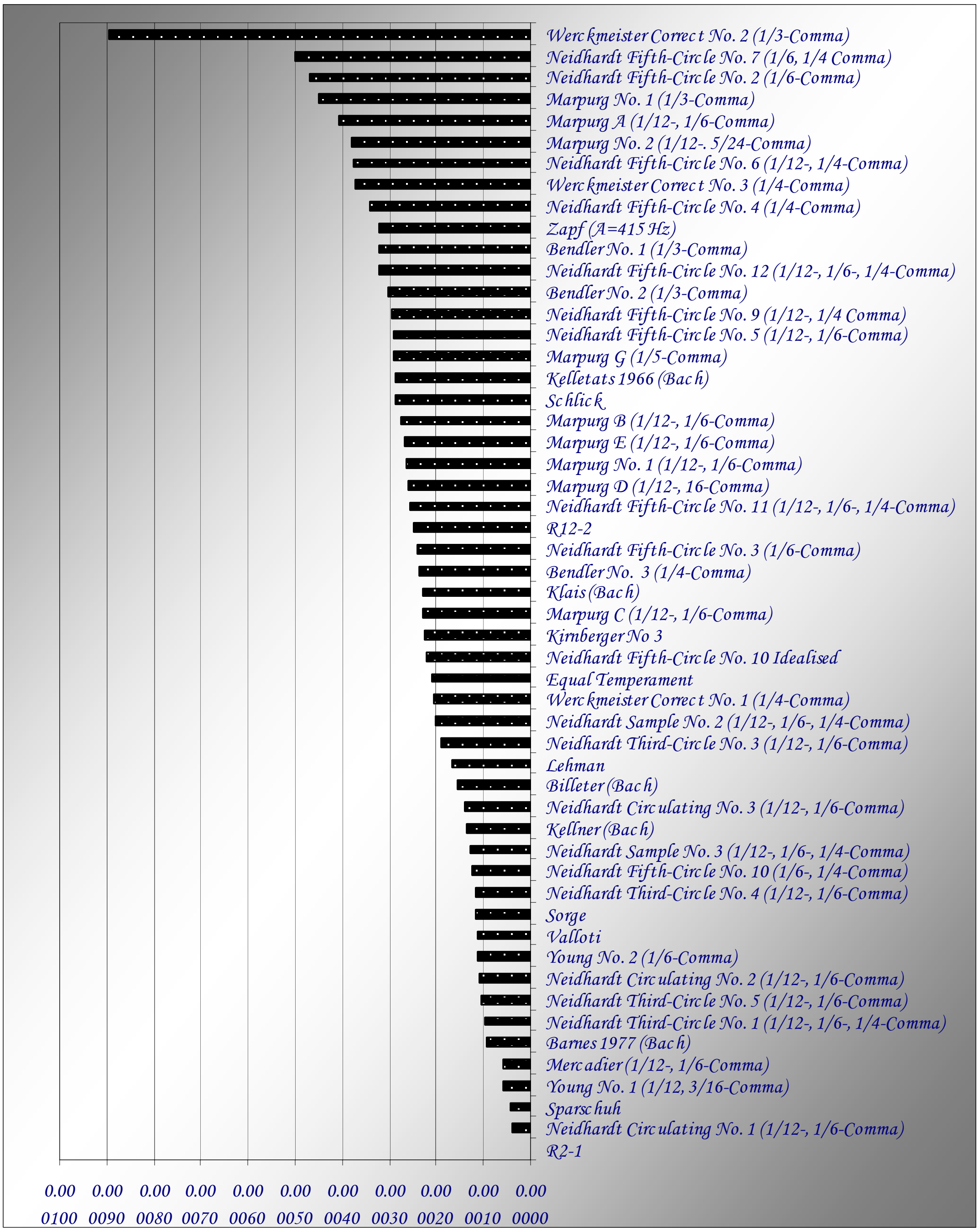


Figure 13: Comparison of R2-1 with other temperaments (correlation distance)

Approximations to the Tuning Ideal

Noting Lindley's remarks concerning the inadequacy of a 1/12-comma Pythagorean tempering unit to capture the nuances of Bach's tuning, it is instructive to consider what happens when Bach's scheme is represented by other fractions of a comma. For the purpose of comparison and completeness, temperaments are presented for 1/11, 1/12, 1/13, 1/14, 1/15, 1/16, 1/17, 1/18 Pythagorean comma fractions, denoted as

R12-11P[39], R12-12P[40], R12-13P, R12-14P, R12-15P, R12-16P, R12-17P and R12-18P, respectively. The results are shown in Table 4.

Cents	C	D♭	D	E♭	E	F	G♭	G	A♭	A	B♭	B
R12-2	0.0	100.6	197.0	301.3	394.8	501.1	598.7	697.8	800.6	895.3	1001.5	1096.7
R12-18P	0.0	103.3	198.7	304.6	397.4	500.7	601.3	699.3	803.9	898.0	1005.2	1099.3
R12-17P	0.0	102.6	198.4	303.8	396.8	500.8	600.7	699.2	803.2	897.6	1004.4	1098.7
R12-16P	0.0	102.0	198.0	302.9	396.1	501.0	600.0	699.0	802.4	897.1	1003.4	1098.0
R12-15P	0.0	101.2	197.7	302.0	395.3	501.2	599.2	698.8	801.6	896.5	1002.3	1097.3
R12-14P	0.0	100.3	197.2	300.8	394.4	501.4	598.3	698.6	800.6	895.8	1001.1	1096.4
R12-13P	0.0	99.2	196.7	299.5	393.4	501.7	597.3	698.3	799.4	895.0	999.7	1095.3
R12-12P	0.0	98.0	196.1	298.0	392.2	502.0	596.1	698.0	798.0	894.1	998.0	1094.1
R12-11P	0.0	96.6	195.4	296.3	390.8	502.3	594.7	697.7	796.4	893.1	996.1	1092.7

Table 4: Cornet-ton temperaments in cents for different fractions of the Pythagorean comma, theoretic ideal R12-14P (green) and realisation based on equal-beating tuning R12-2 (yellow)

Table 5 shows the temperaments that occur when other fractions of a comma are used as approximations to Bach's cammerton tuning. Temperaments are given for 1/11, 1/12, 1/13, 1/14, 1/15, 1/16, 1/17, 1/18 Pythagorean comma fractions, denoted as R2-11P, R2-12P, R2-13P, R2-14P, R2-15P, R2-16P, R2-17P and R2-18P, respectively.

[39] The suffix -nP indicates that the nth fraction of the Pythagorean comma is being considered as the smallest tempering unit.

[40] The 1/12th comma approximation is the Lehman reading.

Cents	C	D♭	D	E♭	E	F	G♭	G	A♭	A	B♭	B
R2-1	0.0	95.2	198.5	299.1	395.5	499.8	593.3	699.6	797.2	896.3	999.1	1093.8
R2-18P	0.0	94.1	194.8	298.0	393.5	499.3	592.2	695.4	796.1	894.1	998.7	1092.8
R2-17P	0.0	94.4	195.6	298.3	394.0	499.4	592.4	696.4	796.3	894.8	998.8	1093.2
R2-16P	0.0	94.6	196.6	298.5	394.6	499.5	592.7	697.6	796.6	895.6	999.0	1093.6
R2-15P	0.0	94.9	197.7	298.8	395.3	499.6	593.0	698.8	796.9	896.5	999.2	1094.1
R2-14P	0.0	95.3	198.9	299.2	396.1	499.7	593.3	700.3	797.2	897.5	999.4	1094.7
R2-13P	0.0	95.6	200.3	299.5	397.0	499.8	593.7	702.0	797.6	898.6	999.7	1095.3
R2-12P[41]	0.0	96.1	202.0	300.0	398.0	500.0	594.1	703.9	798.0	900.0	1000.0	1096.1
R2-11P	0.0	96.6	203.9	300.5	399.3	500.2	594.7	706.2	798.6	901.6	1000.4	1097.0

Table 5: Cammerton temperaments in cents for different fractions of the Pythagorean comma, theoretic ideal R2-14P (green) and realisation based on equal-beating tuning R2-1 (yellow)

The major and minor thirds and fifths can now be compared to see the impact of choosing different comma sizes. Only the cornet-ton case need be considered, as cammerton yields identical results under transposition. Moreover, the comparison is restricted to two of the more extreme variants, the 1/12-comma Lehman solution on the one hand and 1/18-comma on the other. The progression in the quality of the major thirds from best to worst keys is shown in Figure 14; the minor thirds are shown in Figure 15 and the fifths are given in Figure 16. As can be seen, significant distortion occurs with both 1/12 and 1/18-comma solutions, defeating the ideal of smooth, regular, transitions on the circle-of-fifths. In the case of R12-12P reading right to left, the major thirds on F, B♭, E♭, A♭, D♭ increase in size, but at G♭ the third becomes smaller before increasing again at B and finally peaking at E. Comparison with R12-14P shows that this imperfection arises because a 1/14-comma design is being compressed due to a 1/12-comma interpretation. In the minor third case, the behaviour of R12-12P and R12-18P is again arbitrary. The wide-fifth resulting from the 1/12-comma reading in R12-12P is of interest. Jorgensen notes that such harmonic waste is undesirable (see *Tuning*). Again, comparison with R12-14P shows that the imperfection only arises because a 1/14-comma design is being compressed due to a 1/12-comma reading.

[41] This temperament has been mentioned by Lehman, not as a derivation from Bach's diagram, but as a measure to accommodate modern transcriptions.

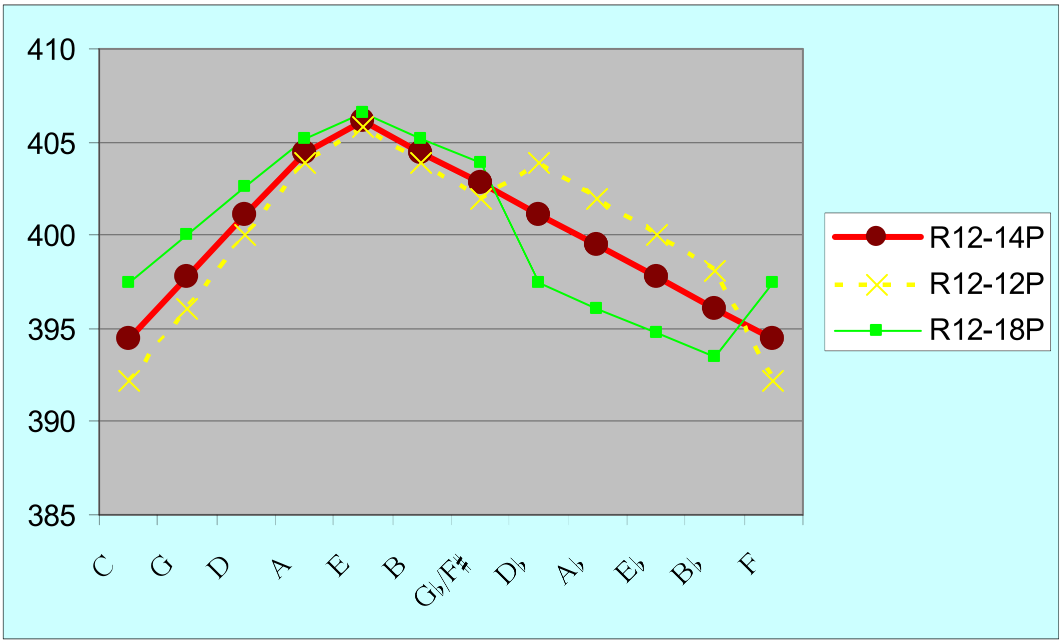


Figure 14: Width of major thirds in R12-14P (theoretic ideal), R12-12P (Lehman) and R12-18P

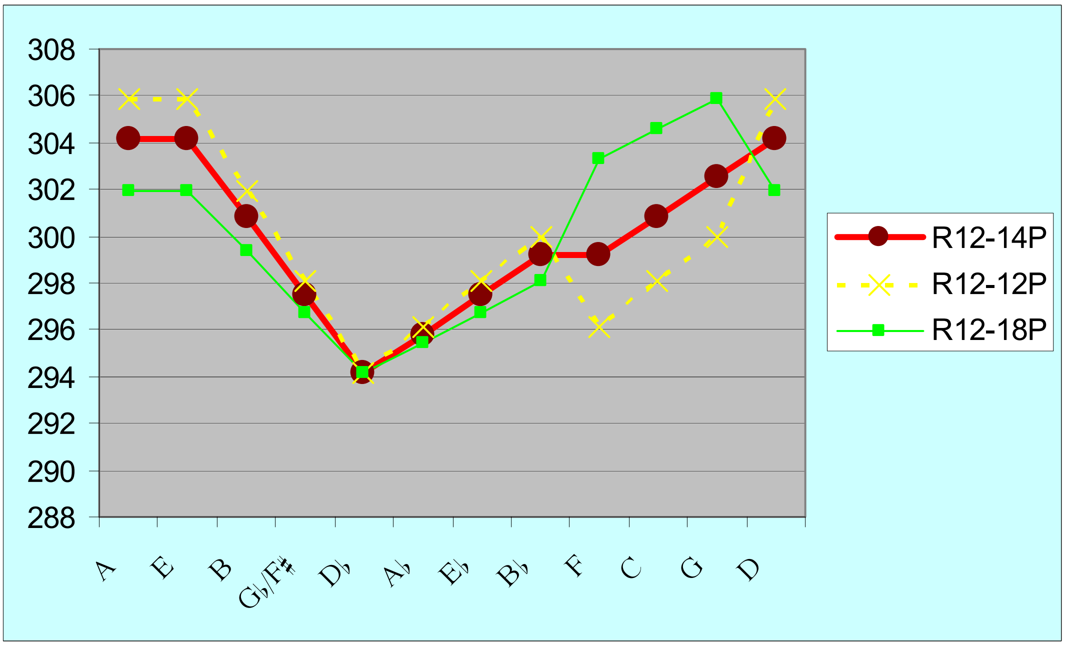


Figure 15: Width of minor thirds in R12-14P (theoretic ideal), R12-12P (Lehman) and R12-18P

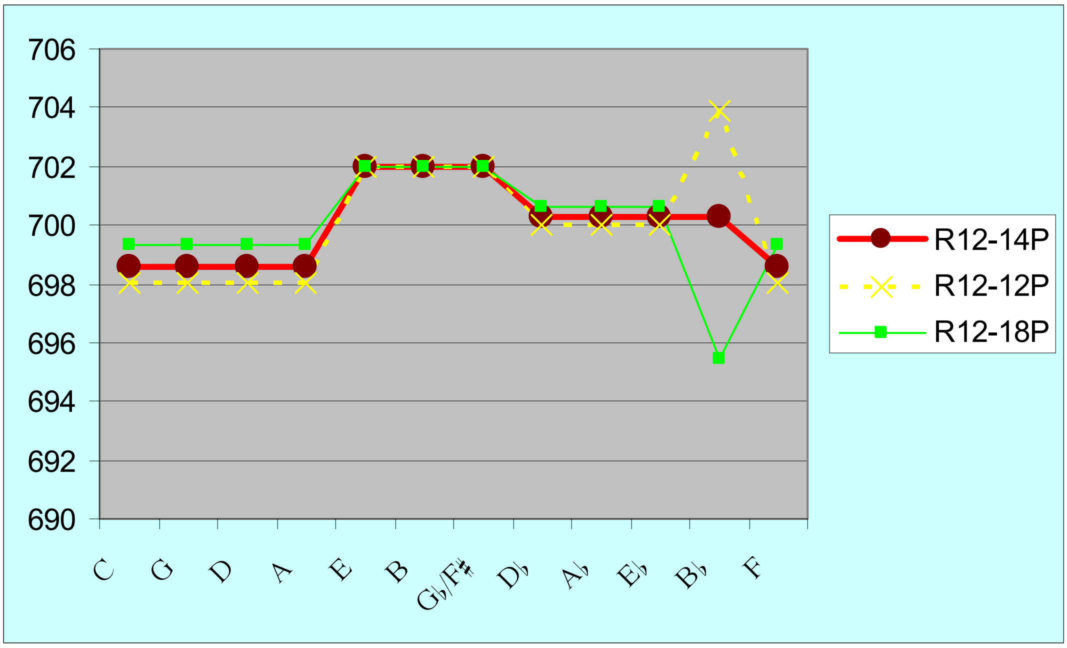


Figure 16: Width of fifths in R12-14P (theoretic ideal), R12-12P (Lehman) and R12-18P

Contending Methods

We have seen that a fourteenth part of a Pythagorean comma is the natural basic tempering unit to theoretically describe Bach's temperament, and that larger fractions compromise the ideal of a progressive gradual change in the size of thirds on the circle-of-fifths, in some cases leading to harmonic waste in the form of a wide fifth. The equal-beating methods R2-1 and R12-2, given by respective left-to-right and right-to-left readings of Bach's diagram, were shown to yield cammerton and cornet-ton transpositions closely approximating the 1/14-comma scheme. The 1/12-comma reading proposed by Lehman was addressed as R12-12P. To complete the picture, the interpretations of Andreas Sparschuh and Michael Zapf will now be considered.

The average minor third must necessarily have a width of 300 cents, the major third 400 cents, and the fifth 700 cents. Improving a third or fifth in one place, must degrade corresponding intervals elsewhere. The standard deviation, which is a measure of the inequality or colour of a temperament, is shown in Figure 17. R12-14P and R2-14P have the least deviation of the thirds and fifths, while R12-12P (Lehman) has the most. The deviation in the fifths, for example, is some 33% greater, perhaps, making it somewhat harder for singers to hit the right note.

The Zapf and Lehman proposals are essentially variants of the cornet-ton solution R12-14P, while the proposal from Sparschuh is a variant of the cammerton solution R2-14P. The comparison in terms of major thirds is shown in Figure 18. Zapf has a best third F-A comparable to R12-14P and a peak A-C#. R2-14P is simply a displaced version of R12-14P, commencing on cammerton D rather than cornet-ton C. Lehman has a greater tempering of C-E and F-A and the widest third E-G#. His reading further improves the already good thirds of F-A, C-A, G-B and D-A, at the expense of D♭-F, A♭-C, E♭-G and B♭-D. The decrease in width through the circle-of-fifths from E-G# to F-A is no longer monotonic, with a problem area at D♭-F and subsequent degraded thirds along the circle-of-fifths until F-A. The temperament of Sparschuh is broadly comparable to R2-14P, although it wanders around somewhat.

Figure 19 shows the corresponding situation for minor thirds. R12-14P has best minor thirds A-C, E-G, and D-F, with a smooth progression toward the narrowest third D♭-E. Zapf has best minor thirds at A-C, with a fairly smooth progression to the narrowest at F#-A. Lehman improves the already good thirds A-C, E-G and D-F at the expense of degrading F-A♭, C-E♭ and G-B♭. His temperament does not exhibit a monotonic rise, as there is an out-of-character drop in width occurring at F-A♭. The temperament of Sparschuh is reasonably well-behaved, although not entirely monotonic in ascent.

The sizes of fifths are shown in Figure 20. R12-14 has three sizes of fifth, while Lehman has four including a wide fifth exhibiting harmonic waste. Both Zapf and Sparschuh use many sizes of fifths.

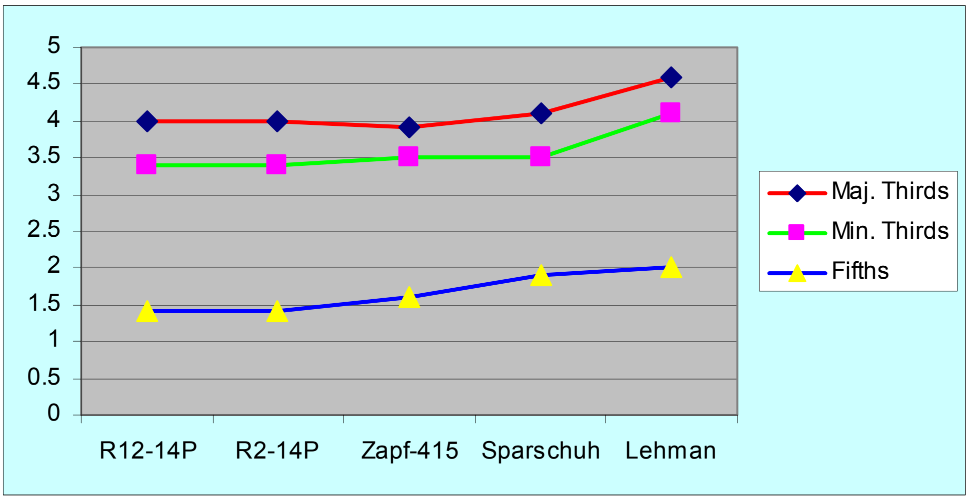


Figure 17: Standard deviation of thirds and fifths (cents)

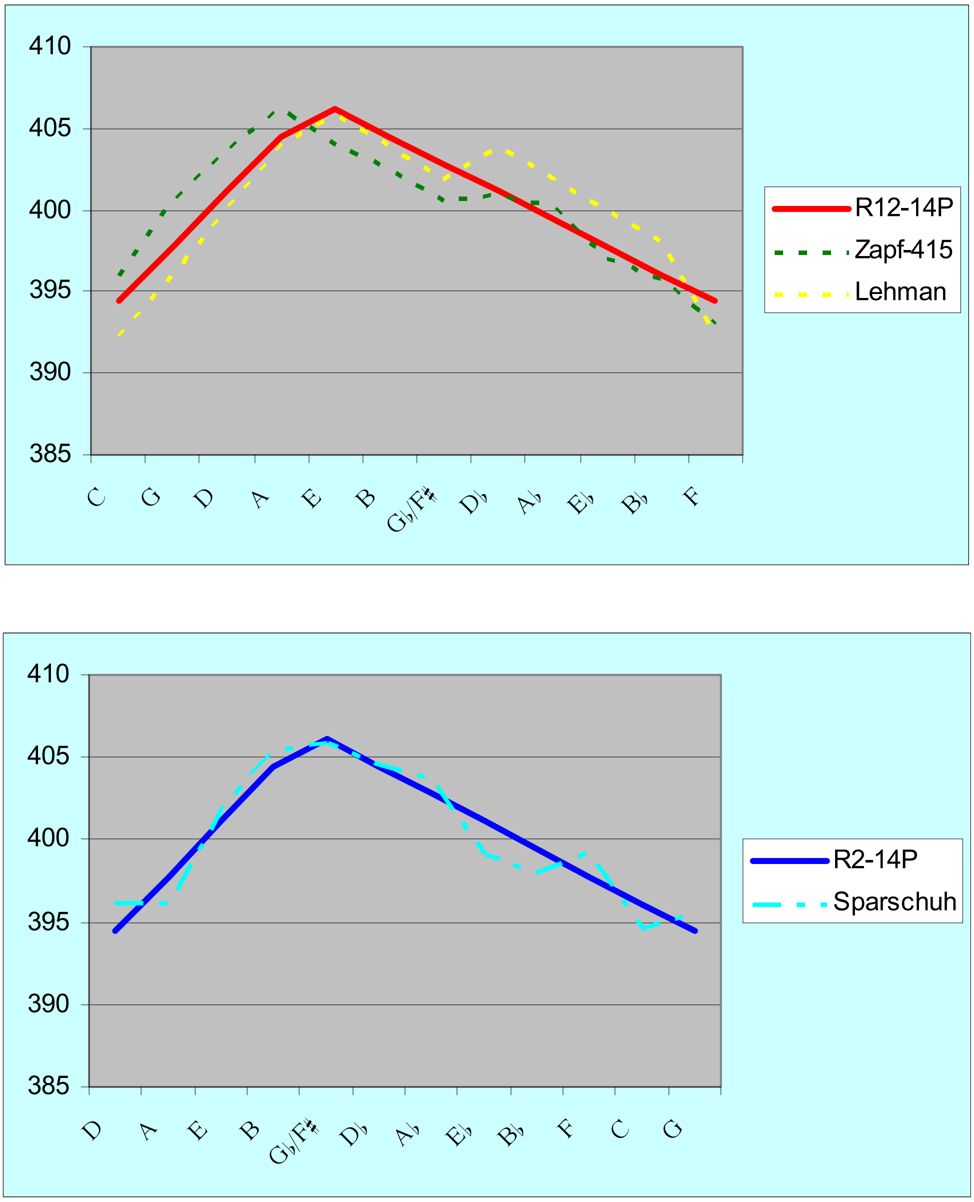
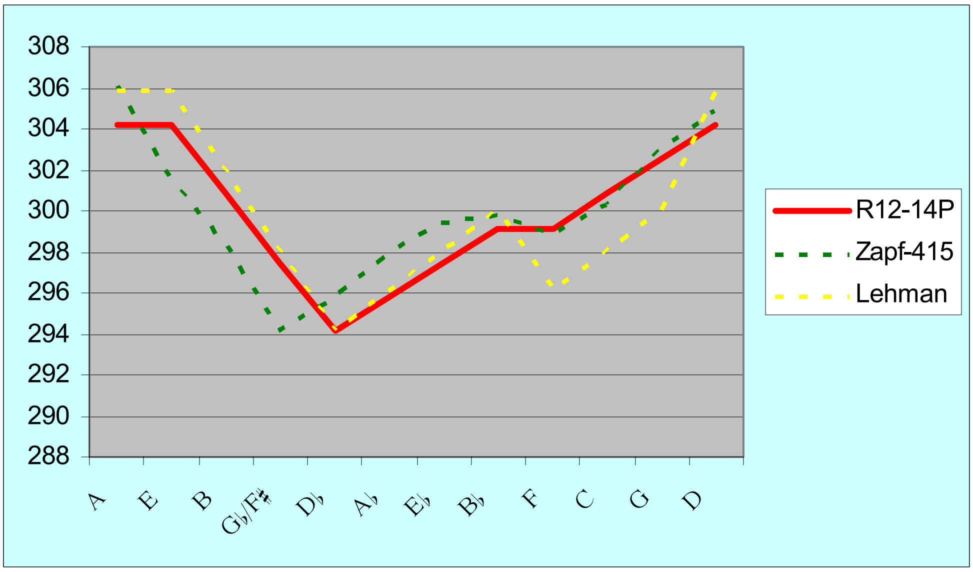


Figure 18: Comparison of major thirds (cents)



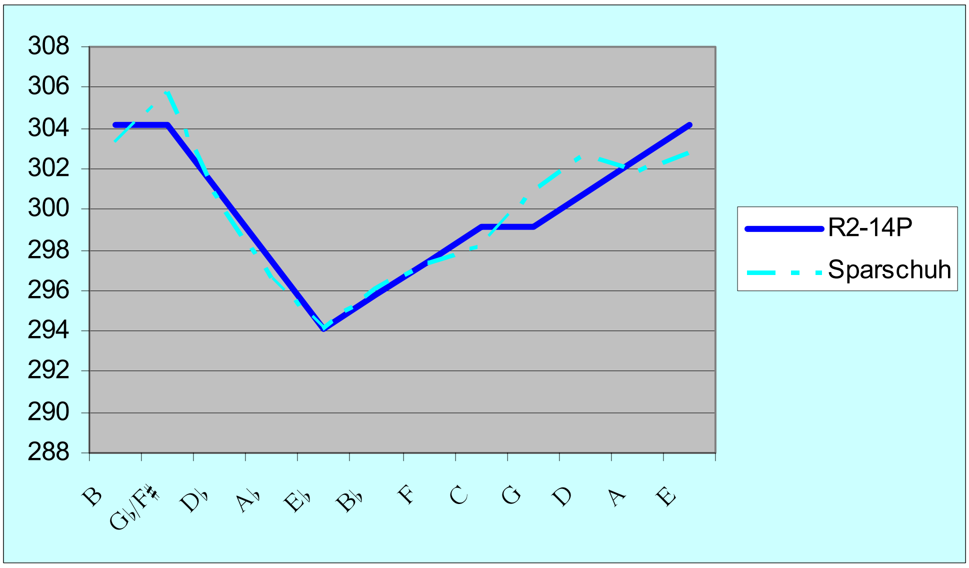


Figure 19: Comparison of minor thirds (cents)

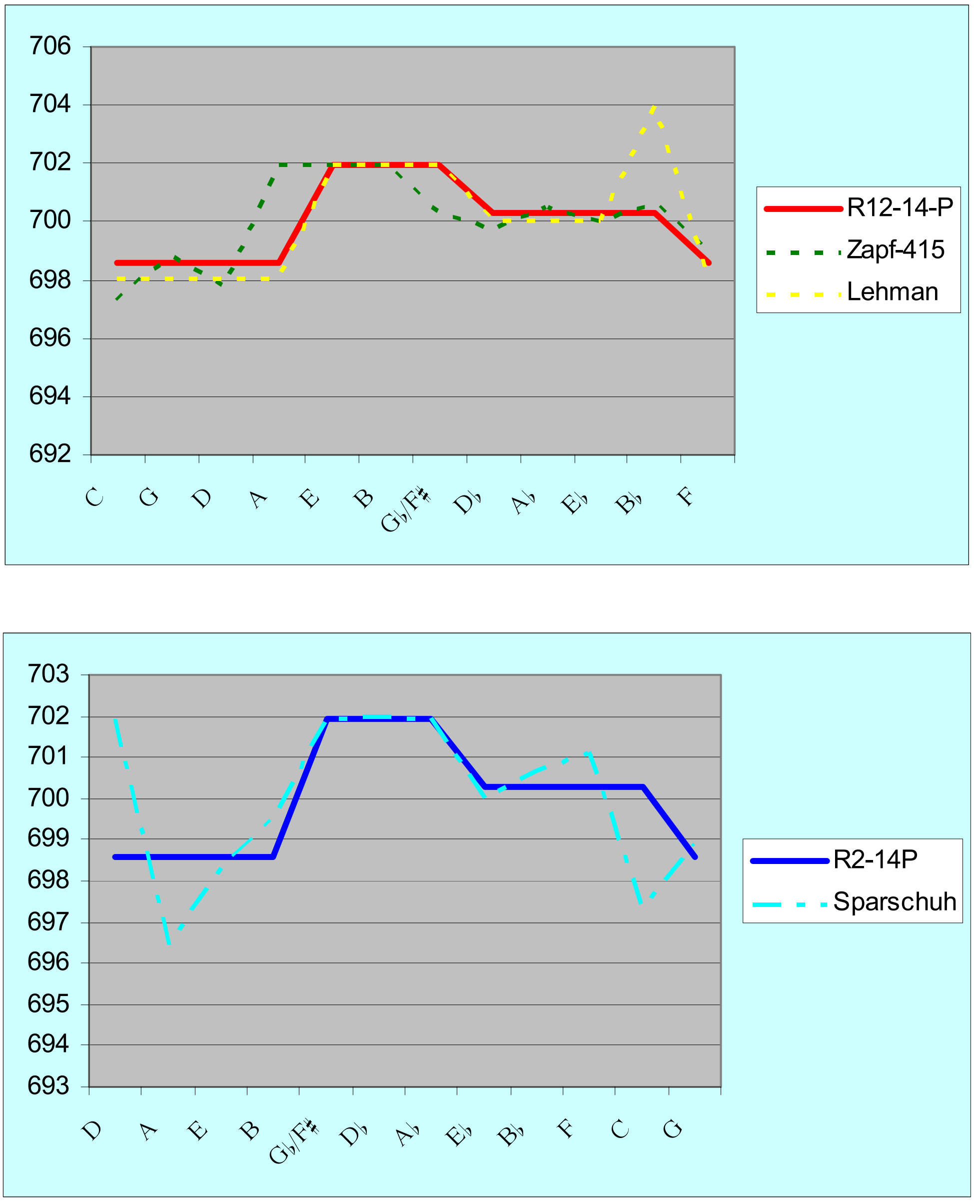


Figure 20: Comparison of fifths (cents)

Conclusions

J. S. Bach's specification for *Das Wohltemperirte Clavier* at the top of his 1722 manuscript is a prescriptive tuning method for an equal-beating temperament. The scheme uses an easy-to-set tempo of 1 beat per second. Read left-to-right, the diagram indicates the cammerton tuning procedure for the octave starting on Middle C. The beat rate of the implicitly-tuned interval closing the circle-of-fifths is shown by the left protrusion of Bach's diagram. Given that stringed instruments need to be regularly tuned, it makes eminent sense that the cammerton tuning should be read from left-to-right and not vice versa. The simple manner by which Bach could determine the relationship between cammerton and cornet-ton beat rates was presented. The cornet-ton tuning also takes place within the octave starting on Middle C, but commences on F, and is based on a right-to-left reading with the implicitly-tuned interval closing the circle-of-fifths at the right. The manner in which cammerton and cornet-ton temperaments have been so combined into one diagram is indeed ingenious.

The rationale behind Bach's temperament was shown to be an optimally smooth progression in the quality of thirds across the circle-of-fifths. It was shown that the tuning specification in terms of beats is closely linked to a 1/14-comma theoretical scheme, and it was demonstrated that the principles behind the design are violated when a 1/12-comma approximation is used. The 1/14-comma scheme was also compared with the readings of Sparschuh and Zapf. Although all are workable musical solutions, only the 1/14-comma reading achieves the ideal of the temperament design.

The pitches obtained by reverse-engineering Bach's diagram accord with our current knowledge of historical pitch[42]. The mathematics shows that the octave being tuned is the one based on Middle C, for both cammerton and cornet-ton cases, and that the respective left-to-right and right-to-left readings commence the tuning sequence on C and F, respectively.

An interesting question remains: should one tune in the theoretical manner of R12-14P and R2-14P, or using the equal-beating temperaments R12-2 and R2-1? With choral music, the theoretic form may, perhaps, facilitate better intonation, as fewer intervals are used and all intervals are exact. On the other hand, it may be that certain works of Bach are more authentically tuned by using the equal-beating procedure encoded in *Das Wohltemperirte Clavier*.

Acknowledgements

Michael Zapf first drew my attention to the important discovery of Andreas Sparschuh, while my interest in Bach's keyboard temperament was stimulated by the enthusiasm and kindness of the late Herbert Anton Kellner. The preparation of this article was facilitated by Capella music software and by Yo Tomita's Bach Musicological Font. Special thanks are due to Thomas Braatz for his invaluable assistance.

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[42] See Bruce Haynes, *A History of Performing Pitch*.

Appendix 1: R2-1 Mathematical Analysis[43]

```
Reduce[{

  (* Pitches f0…f11 form an octave, with relationship on
    the circle-of-fifths according to Bach's WTC diagram read left
    to right *)

  3f0  - 2f7  == 1, (* left end: 5th beating once per second *)
  3f5  - 4f0  == 1, (* loop 1: 4th beating once per second *)
  3f10 - 4f5  == 1, (* loop 2: 4th beating once per second *)
  3f3  - 2f10 == 1, (* loop 3: 5th beating once per second *)
  3f8  - 4f3  == 0, (* loop 4: pure 4th *)
  3f1  - 2f8  == 0, (* loop 5: pure 5th *)
  3f6  - 4f1  == 0, (* loop 6: pure 4th *)
  3f11 - 4f6  == 2, (* loop 7: 4th beating twice per second *)
  3f4  - 2f11 == 2, (* loop 8: 5th beating twice per second *)
  3f9  - 4f4  == 2, (* loop 9: 4th beating twice per second *)
  3f2  - 2f9  == 2, (* loop 10: 5th beating twice per second *)
  3f7  - 4f2  == 2, (* loop 11: 4th beating twice per second *)

   (* Cents corresponding to pitch relations *)

  c0  == 1200Log[2,1.0],
  c1  == 1200Log[2,f1/f0],
  c2  == 1200Log[2,f2/f0],
  c3  == 1200Log[2,f3/f0],
  c4  == 1200Log[2,f4/f0],
  c5  == 1200Log[2,f5/f0],
  c6  == 1200Log[2,f6/f0],
  c7  == 1200Log[2,f7/f0],
  c8  == 1200Log[2,f8/f0],
  c9  == 1200Log[2,f9/f0],
  c10 == 1200Log[2,f10/f0],
  c11 == 1200Log[2,f11/f0]},{c1,c2,c3,c4,c5,c6,c7,c8,c9,c10,c11}]


(* Results c0,…,c11 in cents, and f0,…,f11 in Hz *)

c0==0.&&c1==95.2136&&c10==999.128&&c11==1093.77&&c2==198.495&&c3==299
.124&&c4==395.505&&c5==499.782&&c6==593.259&&c7==699.637&&c8==797.169
&&c9==896.314&&f0==249.072&&f1==263.154&&f10==443.573&&f11==468.497&&
f2==279.331&&f3==296.049&&f4==312.998&&f5==332.43&&f6==350.873&&f7==3
73.109&&f8==394.732&&f9==417.997
```

---

[43] The computer-supported analysis above has been performed with the aid of the symbolic equation solving program, Mathematica (Version 4.2.1). The beat rate of each fourth is given by the difference in the frequency of the 4th harmonic of the lower note and the 3rd harmonic of the higher one, while the beat rate of each fifth is given by the difference in frequency between the 3rd harmonic of the lower note and the 2nd harmonic of the higher one. The tuning sequence starts at f0 moving towards the flats. Fourths and fifths are chosen appropriately so as to remain within the 12 contiguous semitones. The solution shown in the "results" section, shows that f0 is Middle C, implying that the cammerton octave starting on Middle C is tuned starting on C.

Appendix 2: R12-2 Mathematical Analysis[44]

```
Reduce[{

(* Pitches f0...f11 form an octave with relationship on the
 Circle-of-fifths according to Bach's WTC diagram read right to
 left *)

3f10 - 4f5  == 2, (* right end: 4th beating twice per second *)
3f5  - 4f0  == 2, (* loop 11: 4th beating twice per second *)
3f0  - 2f7  == 2, (* loop 10: 5th beating twice per second *)
3f7  - 4f2  == 2, (* loop 9: 4th beating twice per second *)
3f2  - 2f9  == 2, (* loop 8: 5th beating twice per second *)
3f9  - 4f4  == 2, (* loop 7: 4th beating twice per second *)
3f4  - 2f11 == 0, (* loop 6: pure 5th *)
3f11 - 4f6  == 0, (* loop 5: pure 4th *)
3f6  - 4f1  == 0, (* loop 4: pure 4th *)
3f1  - 2f8  == 1, (* loop 3: 5th beating once per second *)
3f8  - 4f3  == 1, (* loop 2: 4th beating once per second *)
3f3  - 2f10 == 1, (* loop 1: 5th beating once per second *)

(* Cents corresponding to pitch relations *)

c0  == 1200Log[2,1.0],
c1  == 1200Log[2,f1/f0],
c2  == 1200Log[2,f2/f0],
c3  == 1200Log[2,f3/f0],
c4  == 1200Log[2,f4/f0],
c5  == 1200Log[2,f5/f0],
c6  == 1200Log[2,f6/f0],
c7  == 1200Log[2,f7/f0],
c8  == 1200Log[2,f8/f0],
c9  == 1200Log[2,f9/f0],
c10 == 1200Log[2,f10/f0],
c11 == 1200Log[2,f11/f0]},{c1,c2,c3,c4,c5,c6,c7,c8,c9,c10,c11}]


(* Results c0,…,c11 in cents, and f0,…,f11 in Hz *)

c0==0.&&c1==100.628&&c10==1001.5&&c11==1096.72&&c2==197.01&&c3==301.2
86&&c4==394.763&&c5==501.141&&c6==598.673&&c7==697.818&&c8==800.633&&
c9==895.273&&f0==279.331&&f1==296.049&&f10==498.145&&f11==526.309&&f2
==312.998&&f3==332.43&&f4==350.873&&f5==373.109&&f6==394.732&&f7==417
.997&&f8==443.573&&f9==468.49
```

---

[44] The computer-supported analysis above has been performed with the aid of the symbolic equation solving program, Mathematica (Version 4.2.1). The beat rate of each fourth is given by the difference in the frequency of the 4th harmonic of the lower note and the 3rd harmonic of the higher one, while the beat rate of each fifth is given by the difference in frequency between the 3rd harmonic of the lower note and the 2nd harmonic of the higher one. The tuning sequence starts at f0 moving towards the sharps. Fourths and fifths are chosen appropriately so as to remain within the 12 contiguous semitones. The solution shown in the "results" section, shows that f0 is Middle C, implying that the cornet-ton octave starting on Middle C is tuned starting on F.

